Available online at www.sciencedirect.com
SCIENCE DIRECT
ACADEMIC PRESS
Journal of Sound and Vibration 265 (2003) 887–896
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Missing frequencies in previous exact solutions of free vibrations of simply supported rectangular plates

R.C. Batra*, S. Aimmanee

Department of Engineering Science and Mechanics, M/C 0219, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Received 28 October 2002; accepted 4 November 2002

1. Introduction

Srinivas and Rao [1] and Srinivas et al. [2] have studied free vibrations of homogeneous isotropic and orthotropic simply supported rectangular plates. Their work has been extended by Heyliger and Saravanos [3] to free vibrations of a simply supported hybrid piezoelectric plate, and by Batra et al. [4,5] to simply supported laminated plates made of a non-piezoelectric lamina either surrounding thin piezoelectric layers or enclosed between them. The key feature of these analyses is that the three components of displacement are expanded in terms of double Fourier series in the plane of the plate. The coefficient of each term in the double Fourier series is taken as a function of the thickness co-ordinate only. These expansions are such that boundary conditions are exactly satisfied at the edges of the simply supported plate. In all analyses known to the authors, some solutions involving null transverse displacements have been neglected thereby tacitly ignoring some of the in-plane modes of vibration. The in-plane modes of vibration may be important in free vibrations of thick plates for which frequencies of in-plane modes of vibration are lower than those of the lateral modes of vibration.

2. Analysis of the problem

We use rectangular Cartesian co-ordinates to describe three-dimensional deformations of a rectangular plate of thickness h and sides of length L_1 and L_2 along the x_1- and x_2-axis, respectively, with the plate occupying the region $0 \leqslant x_1 \leqslant L_1$, $0 \leqslant x_2 \leqslant L_2$, $0 \leqslant x_3 \leqslant h$. Assuming that the plate is made of a homogeneous linear elastic material, equations governing its free vibrations are

$$C_{ijkl} u_{l,kj} + \rho\omega^2 u_i = 0, \quad i = 1, 2, 3, \tag{1}$$

*Corresponding author. Tel.: +1-540-2316051; fax: +1-540-2314574.
E-mail address: rbatra@vt.edu (R.C. Batra).

0022-460X/03/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0022-460X(02)01568-7

where $\mathbf{C}$ is the elasticity tensor satisfying $C_{ijkl} = C_{klij} = C_{ijlk}$, $\mathbf{u}$ the displacement of a point, ρ the mass density, ω the frequency of free vibration, $u_{i,j} = \partial u_i/\partial x_j$, $\mathbf{x}$ the position of a material particle in the reference configuration and a repeated index implies summation over the range of the index.

2.1. Simply supported plates

Commonly used boundary conditions for free vibrations of a simply supported plate are

$$\begin{aligned} &u_2 = u_3 = 0, \ \ \sigma_{11} = C_{11kl}u_{l,k} = 0 \quad \text{on } x_1 = 0, L_1, \\ &u_1 = u_3 = 0, \ \ \sigma_{22} = C_{22kl}u_{l,k} = 0 \quad \text{on } x_2 = 0, L_2, \\ &\sigma_{i3} = C_{i3kl}u_{l,k} = 0 \quad \text{on } x_3 = 0, h. \end{aligned} \tag{2}$$

Here, $\sigma_{ij} = C_{ijkl}u_{l,k}$ is the stress tensor. Following Srinivas and Rao [1], we assume that

$$\begin{aligned} u_1 &= \sum_{m,n=0}^{\infty} U_1^{mn}(x_3)\cos\frac{m\pi x_1}{L_1}\sin\frac{n\pi x_2}{L_2}, \\ u_2 &= \sum_{m,n=0}^{\infty} U_2^{mn}(x_3)\sin\frac{m\pi x_1}{L_1}\cos\frac{n\pi x_2}{L_2}, \\ u_3 &= \sum_{m,n=0}^{\infty} U_3^{mn}(x_3)\sin\frac{m\pi x_1}{L_1}\sin\frac{n\pi x_2}{L_2}. \end{aligned} \tag{3}$$

The displacement field (3) identically satisfies boundary conditions $(2)_1$ and $(2)_2$ on the plate edges. Thus, only boundary conditions $(2)_3$ on the top and the bottom surfaces of the plate and Eqs. (1) need to be satisfied.

The difference between Eq. (3) and the form assumed by Srinivas and Rao [1] is that we allow for the possibility of either $m = 0$ or $n = 0$ and they do not. For $m = n = 0$, all displacement components vanish identically and we have a null solution. However, when either m or n is positive and the other is zero, we get in-plane pure distortional (i.e., the dilatation $u_{i,i} = 0$) modes of vibration neglected in the earlier studies.

Since previous works considered the case of $m \geqslant 1$ and $n \geqslant 1$, we focus on the case when either m or n equals zero. Let $m = 0$. Then, Eqs. (3) become

$$u_1 = \sum_{n=1}^{\infty} U_1^{n}(x_3)\sin\frac{n\pi x_2}{L_2}, \quad u_2 = 0, \quad u_3 = 0. \tag{4}$$

We analyze the problem for the plate material exhibiting different symmetries.

2.1.1. Plate made of an isotropic material

For an isotropic material,

$$C_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5}$$

where δ_{ij} is the Kronecker delta, E Young's modulus and v the Poisson ratio. The satisfaction of governing equations (1) and boundary conditions $(2)_3$ on $x_3 = 0$ and h requires that $U_1^n(x_3)$ be independent of x_3, and

$$\omega = \sqrt{\frac{E}{2(1+v)\rho}}\frac{n\pi}{L_2}, \quad n = 1, 2, \dots \quad (6)$$

These are natural frequencies of modes

$$u_1 = U_1^n \sin\frac{n\pi x_2}{L_2}, \quad u_2 = u_3 = 0, \quad n = 1, 2, \dots \quad (7)$$

Similarly,

$$\omega = \sqrt{\frac{E}{2(1+v)\rho}}\frac{m\pi}{L_1}, \quad m = 1, 2, \dots, \quad (8)$$

are natural frequencies for mode shapes:

$$u_1 = 0, \;\; u_2 = U_2^m \sin\frac{m\pi x_1}{L_1}, \;\; u_3 = 0, \quad m = 1, 2, \dots \quad (9)$$

Note that frequencies (6) and (8) are independent of the thickness of the plate, and depend only on the shear modulus $G = E/2(1+v)$, the mass density and lengths of the two sides of the rectangular plate; the dependence upon the shear modulus and the mass density is through the speed of the shear wave. The corresponding displacement fields induce only the in-plane shear stress σ_{12} and all fields are uniform through the plate thickness.

For a square plate, frequencies given by Eqs. (6) and (8) coincide and the corresponding mode shape is

$$u_1 = U_1^n \sin\frac{n\pi x_2}{L}, \quad u_2 = U_2^n \sin\frac{n\pi x_1}{L}, \quad u_3 = 0, \quad (10)$$

where L is the length of a side of the plate.

For a square plate with $L = 5h = 20$ and $v = 0.3$, non-dimensional flexural frequencies listed in Table 1 of Ref. [2] for the antisymmetric mode IA of vibration are

$$\lambda_{1,1} = 0.34207, \quad \lambda_{2,1} = 0.75111, \quad \lambda_{2,2} = 1.0889, \quad (11)$$

where $\lambda = \omega h\sqrt{\rho/G}$, and subscripts 1, 1 on λ signify that $m = 1$, $n = 1$ in Eqs. (3). For $m = 0$ and $n = 1$ or $m = 1$ and $n = 0$, the frequency $\lambda_{1,0} = \lambda_{0,1}$ of the in-plane vibration mode computed from Eq. (6) or (8) equals 0.6284 which is smaller than $\lambda_{2,1}$. Thus, the second lowest frequency is $\lambda_{1,0}$ and not $\lambda_{2,1}$.

2.1.2. Laminated isotropic plates

For a laminated plate made of two or more isotropic homogeneous lamina perfectly bonded together, modes (7) and (9) with the corresponding frequencies (6) and (8) are feasible only if the speed of the shear wave in each lamina is the same. Otherwise, the requirements of the displacements at an interface between two adjoining laminae rules out these modes of vibration.

2.1.3. Plate made of a transversely isotropic material

When the axis of transverse isotropy is along the x_3-axis, then the material is isotropic in the x_1–x_2 plane. Therefore, mode shapes (7) and (9) with frequencies (6) and (8), respectively, are admissible with the difference that the shear modulus G is replaced by the in-plane shear modulus G_{12}. For the case of the axis of transverse isotropy making an angle θ with the x_3-axis, mode shapes (7) are inadmissible except when $\theta = 90^\circ$. For $\theta = 90^\circ$,

$$\omega_n = (G_{13}/\rho)^{1/2}\frac{n\pi}{L_2}, \quad n = 1, 2, \ldots \, . \tag{12}$$

2.1.4. Plates made of orthotropic materials

For 0° and 90° homogeneous orthotropic plates, the principal axes of the material will be parallel to the edges of the rectangular plate, and also to the co-ordinate axes. Mode shapes (7) and (9) with the corresponding frequencies $(G_{12}/\rho)^{1/2}n\pi/L_2$ and $(G_{12}/\rho)^{1/2}m\pi/L_1$ are admissible. Here G_{12} is the shear modulus in the x_1–x_2 plane of the plate. When the material principal axes in the x_1–x_2 plane make an angle θ with the co-ordinate axes, then frequencies of mode shapes (7) and (9) are inadmissible.

For a rectangular plate made of Aragonite crystal with material properties listed in Table 1 of [1] and $L_1 = 2L_2 = 5h = 20$, $\lambda = \omega h\sqrt{(\rho/E_{11})}$, we get from Table 5 of [1], $\lambda_{1,1} = 0.29690$, $\lambda_{2,1} = 0.51342$ for mode *IA* of flexural vibration. Frequencies of in-plane modes of vibration are $\lambda_{1,0} = 0.32231$, $\lambda_{0,1} = 0.64462$. Thus, $\lambda_{1,0} < \lambda_{2,1}$, and the second lowest frequency is $\lambda_{1,0}$ and not $\lambda_{2,1}$.

2.1.5. Laminated orthotropic plates

As for laminated isotropic plates, for a laminated plate made of two or more either 0° or 90° orthotropic homogeneous lamina perfectly bonded together with co-ordinate axes parallel to the material principal axes in each lamina, modes (7) and (9) with the corresponding frequencies listed above are admissible only if the speeds of shear waves of the same amplitude and propagating in the same direction in the two adjoining laminae are equal. Otherwise, these displacements are inadmissible.

2.1.6. Functionally graded plates

For a functionally graded plate with material properties varying in the x_3-direction only, vibration modes (7) and (9) are admissible only if G/ρ is independent of x_3; the corresponding frequencies for an isotropic, transversely isotropic and orthotropic plate are given by the pertinent equations listed in the corresponding subsections.

2.1.7. Plate made of a monoclinic material

Consider a rectangular plate made of a homogeneous linear elastic monoclinic material with the x_1x_2-plane as the single plane of material symmetry and co-ordinate axes aligned along the material principal axes. Hooke's law for this material is [6]

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ (u_{3,2}+u_{2,3}) \\ (u_{1,3}+u_{3,1}) \\ (u_{2,1}+u_{1,2}) \end{Bmatrix}. \tag{13}$$

The $c_{ik} = c_{ki}$ are elastic constants. The stress field obtained by substituting displacement fields (7) or (9) into Eq. (13) does not satisfy the balance of linear momentum (1). Hence, in-plane modes of vibration (7) and (9) are inadmissible in this monoclinic plate irrespective of boundary conditions at the edges. When the plane of material symmetry is the x_1–x_3 plane, then the matrix of elastic constants is

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ c_{21} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{31} & c_{32} & c_{33} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & c_{44} & 0 & c_{46} \\ c_{15} & c_{25} & c_{35} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & c_{46} & 0 & c_{66} \end{bmatrix} \tag{14}$$

and boundary conditions $(2)_3$ on the top and the bottom surfaces make displacements (7) and (9) inadmissible.

2.1.8. Plate made of other anisotropic materials

Chadwick et al. [7] have shown that there are exactly eight different sets of symmetry planes admissible for the fourth order elasticity tensor **C**. Besides the isotropic, transversely isotropic, orthotropic and the monoclinic materials studied above, we analyze the other four materials below. The displacement fields (7) and (9) are not possible modes of free vibration in a plate made of an anisotropic material (or a triclinic material) with 21 elastic constants and a trigonal material. However, they are possible in a tetragonal material and a cubic material for which the matrix of elastic constants, respectively, are

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \quad \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}. \tag{15}$$

Frequencies corresponding to modes (7) and (9) are given by $\sqrt{(c_{66}/\rho)}n\pi/L_2$ and $\sqrt{(c_{44}/\rho)}m\pi/L_1$. The co-ordinate axes have been properly chosen for the matrix of elastic constants to have the forms (15).

2.1.9. Piezoelectric plates

Free vibrations of a simply supported rectangular piezoelectric plate are governed by

$$C_{ijkl}u_{l,kj} + e_{ijk}\phi_{,kj} + \rho\omega^2 u_i = 0,$$
$$e_{kij}u_{j,ik} - \epsilon_{ij}\phi_{,ij} = 0, \qquad (16)$$

where $e_{ijk} = e_{ikj}$ are the piezoelectric constants, $C_{ijkl} = C_{klij} = C_{ijlk}$ are the elastic constants, and $\epsilon_{ij} = \epsilon_{ji}$ are the dielectric constants. Boundary conditions for all bounding surfaces of the plate grounded are

$$u_2 = u_3 = 0,\ \ \sigma_{11} = C_{11kl}u_{l,k} + e_{11k}\phi_{,k} = 0,\ \ \phi = 0 \quad \text{on } x_1 = 0,\ L_1,$$
$$u_1 = u_3 = 0,\ \ \sigma_{22} = C_{22kl}u_{l,k} + e_{22k}\phi_{,k} = 0,\ \ \phi = 0 \quad \text{on } x_2 = 0,\ L_2,$$
$$\sigma_{i3} = C_{i3kl}u_{l,k} + e_{i3k}\phi_{,k} = 0,\ \ \phi = 0 \quad \text{on } x_3 = 0,\ h. \qquad (17)$$

Displacement fields (7) and (9) with $\phi \equiv 0$ satisfy Eqs. (16) and (17) for a transversely isotropic piezoelectric plate with the axis of transverse isotropy along the x_3-axis, and for an orthotropic piezoelectric plate with the material principal axes coincident with the co-ordinate axes. Thus, a simply supported piezoelectric plate with all bounding surfaces grounded can vibrate with frequencies given by Eqs. (6) or (8). For the square PZT plate with $L = 4h = 0.04$ m studied by Heyliger and Savaranos [3],

$$\omega = \sqrt{\frac{30.6 \times 10^9}{1}}\frac{\pi}{0.04} = 1.37366 \times 10^7 \text{ rad/s}, \qquad (18)$$

which is between 0.969299×10^7 and 1.94255×10^7 rad/s listed in Table II of Ref. [3] for the first two flexural modes of vibration. For $L = h = 0.01$ m, Eq. (8) gives $\omega = 5.4955 \times 10^7$ rad/s which is less than the frequency 7.13061×10^7 rad/s of the first flexural mode computed by Heyliger and Savaranos [3]. These authors also computed frequencies when the top and the bottom surfaces are electrically insulated. Frequencies in this case were at least equal to those found for the case of grounded top and bottom surfaces. Displacement fields (7) and (9) with $\phi = 0$ satisfy the boundary condition of electrically insulated major surfaces.

Modes (7) and (9) are admissible in hybrid plates provided that the speeds of shear waves of the same amplitude and propagating in the same direction in the adjoining layers are equal.

2.1.10. Plate theories

As summarized, amongst others, by Cheng and Batra [8], the displacement field for the CPT, the FSDT [9] and the TSDT [10] can be written as

$$u_\alpha(x_i, t) = u^0_\alpha(x_\beta, t) - x_3 u^0_{3,\alpha} + g(x_3)\varphi_\alpha(x_\beta, t),\ \ u_3(x_i, t) = u^0_3(x_\beta, t), \quad \alpha, \beta = 1, 2, \qquad (19)$$

where u^0_α, u^0_3 and φ_γ are independent of x_3 and the function $g(x_3) = 0$ for the CPT, $g(x_3) = x_3$ for the FSDT and $g(x_3) = x_3(1 - 4x_3^2/3h^2)$ for the TSDT. In this subsection, Latin indices range from 1 to 3 and Greek indices from 1 to 2. Function $\mathbf{u}^0$ gives displacements of a point on the midsurface of the plate and for the FSDT $(\varphi_1 - u^0_{3,1})$ and $(-\varphi_2 + u^0_{3,2})$ are, respectively, rotations of the transverse normal to the midsurface about the x_2- and x_1-axis. For vibration modes (7), we set $u^0_3 = \varphi_1 = \varphi_2 = 0$. The relevant plate equation for determining the in-plane modes of vibration is

$$N_{\alpha\beta,\beta} + \omega^2 I_0 u_\alpha = 0, \quad \alpha = 1, 2, \tag{20}$$

where

$$N_{\alpha\beta} = \int_0^h \sigma_{\alpha\beta}\,\mathrm{d}x_3, \quad I_0 = \int_0^h \rho\,\mathrm{d}x_3,$$
$$\sigma_{\alpha\beta} = H_{\alpha\beta\lambda\mu}(u_{\lambda,\mu} + u_{\mu,\lambda})/2, \tag{21}$$

and $H_{\alpha\beta\lambda\mu}$ are the reduced elasticities obtained by incorporating the constraint $\sigma_{33} = 0$ into the constitutive relation. For an isotropic material,

$$H_{\alpha\beta\lambda\mu} = \frac{vE}{1-v^2}\delta_{\alpha\beta}\delta_{\lambda\mu} + \frac{E}{2(1+v)}(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda}). \tag{22}$$

The boundary conditions are

$$u_2 = 0,\ N_{11} = 0 \quad \text{on } x_1 = 0,\ L_1,$$
$$u_1 = 0,\ N_{22} = 0 \quad \text{on } x_2 = 0,\ L_2. \tag{23}$$

For a homogeneous isotropic plate, the displacement field (7) satisfies Eqs. (20) and boundary conditions (23) with ω given by Eq. (6). Thus, any of these three plate theories as well as several other plate theories including the first and the higher order shear and normal deformable plate theories of Vidoli and Batra [11], and Batra and Vidoli [12] will predict frequencies (8) correctly for isotropic as well as orthotropic homogeneous plates. Batra et al. [13] used their plate theory to study plane waves and free vibrations of a plate. They, like other investigators, missed vibration modes (7) and (9) since they also set $m = n = 1$ as the lower limit in summations on the right-hand sides of Eqs. (3).

A thin-plate theory neglects u^0_α and thus will not predict vibration modes (7) and (9).

2.2. Clamped plates

Displacement fields (7) and (9) do not satisfy conditions

$$u_1 = u_2 = u_3 = 0 \quad \text{on } x_1 = 0,\ L_1 \text{ and } x_2 = 0,\ L_2. \tag{24}$$

Therefore, the in-plane modes of vibration (7) and (9) with frequencies given, respectively, by Eqs. (6) and (8) do not occur in a rectangular plate with all four edges clamped.

2.3. Clamped-simply supported plates

When $m = 0$, then the mode shape (7) is admissible in a rectangular plate with edges $x_2 = 0$ and $x_2 = L_2$ clamped and the other two edges simply supported; a similar mode shape occurs when edges $x_1 = 0$ and $x_1 = L_1$ are clamped and the other two edges are simply supported.

Table 1
First few non-dimensional natural frequencies, λ, of a simply supported rectangular plate of different aspect ratios, numbers in parentheses indicate values of (m,n). Frequencies of in-plane modes of vibration missing in previous solutions are in bold

	Square plate						Rectangular plate with $L_1 = L_2$					
No.	$L_1/h = 2$	4	6	8	10	20	2	4	6	8	10	20
1	1.5158	0.5066	0.2457	0.143	0.0932	0.0239	**0.7854**	0.3349	0.1581	0.091	0.0589	0.015
	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	**(0,1)**	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
2	**1.5708**	**0.7854**	**0.5236**	0.3349	0.2226	0.0589	1.0692	**0.3927**	0.2455	0.143	0.0931	0.0239
	(1,0),(0,1)	**(1,0),(0,1)**	**(1,0),(0,1)**	(1,2),(2,1)	(1,2),(2,1)	(1,2),(2,1)	(1,1)	**(0,1)**	(1,2)	(1,2)	(1,2)	(1,2)
3	2.2214	1.0692	0.5544	**0.3927**	**0.3142**	0.0932	1.5158	0.5066	**0.2618**	**0.1963**	0.1485	0.0386
	(1,1)	(1,2),(2,1)	(1,2),(2,1)	**(1,0),(0,1)**	**(1,0),(0,1)**	(2,2)	(1,2)	(1,2)	**(0,1)**	**(0,1)**	(1,3)	(1,3)
4	2.8066	1.1107	0.7405	0.5066	0.3421	0.1155	**1.5708**	0.7606	0.3811	0.226	**0.1571**	0.0502
	(1,2),(2,1)	(1,1)	(1,1)	(2,2)	(2,2)	(1,3),(3,1)	**(1,0),(0,2)**	(1,3)	(1,3)	(1,3)	**(0,1)**	(2,1)
5	**3.1416**	1.5158	0.818	0.5554	0.4171	0.1485	1.7562	**0.7854**	0.4822	0.289	0.1913	0.0589
	(2,0),(0,2)	(2,2)	(2,2)	(1,1)	(1,3),(3,1)	(2,3),(3,2)	(1,1)	**(1,0),(0,2)**	(2,1)	(2,1)	(2,1)	(2,2),(1,4)
6	3.5124	**1.5708**	0.9761	0.6124	0.4445	**0.1571**	2.1219	0.8781	**0.5236**	0.3349	0.2226	0.0733
	(1,2),(2,1)	**(2,0),(0,2)**	(1,3),(3,1)	(1,3),(3,1)	(1,1)	**(1,0),(0,1)**	(1,3)	(1,1)	**(1,0),(0,2)**	(1,4),(2,2)	(2,2),(1,4)	(2,3)
7	3.5306	1.7562	**1.0472**	0.7606	0.5239	0.1913	2.2214	0.9425	0.5544	**0.3927**	0.2735	**0.0785**
	(1,1)	(1,3),(3,1)	**(2,0),(0,2)**	(2,3),(3,2)	(2,3),(3,2)	(1,4),(4,1)	(1,2)	(2,1)	(2,2),(1,4)	**(1,0),(0,2)**	(2,3)	**(0,1)**
8	3.7419	1.7744	1.1708	**0.7854**	**0.6283**	0.2018	**2.3562**	1.0692	0.5854	0.4085	0.313	0.0847
	(2,2)	(1,3),(3,1)	(1,2),(2,1)	**(2,0),(0,2)**	**(2,0),(0,2)**	(3,3)	**(0,3)**	(2,2),(1,4)	(1,1)	(2,3)	(1,5)	(1,5)
9	3.8477	1.8577	1.1934	0.8781	0.6571	0.2221	2.5305	1.1107	0.6686	0.4391	**0.3142**	0.0931
	(1,1)	(1,1)	(2,3),(3,2)	(1,2),(2,1)	(1,4),(4,1)	(1,1)	(2,1)	(1,2)	(2,3)	(1,1)	**(1,0),(0,2)**	(2,4)
10	4.2658	2.1219	1.2461	0.9365	0.6889	**0.3142**	2.8066	**1.1781**	0.7405	0.4652	0.3421	0.1072
	(1,3),(3,1)	(2,2)	(1,1)	(1,1)	(3,3)	**(2,0),(0,2)**	(2,2),(1,4)	**(0,3)**	(1,2)	(1,5)	(2,4)	(3,1)
11	4.4429	2.2214	1.4546	0.9425	0.7025	0.3513	2.8318	1.2656	0.7554	0.5066	0.3512	0.1183
	(2,2)	(2,2)	(1,4),(4,1)	(1,4),(4,1)	(1,2),(2,1)	(1,2),(2,1)	(1,3)	(2,3)	(1,5)	(2,4)	(1,1)	(2,5)
12	4.7124	**2.3562**	1.4812	0.98563	0.7498	0.3755	2.8757	1.4117				**0.1571**
	(1,2),(2,1)	**(3,0),(0,3)**	(2,2)	(3,3)	(1,1)	(1,1)	(1,1)	(1,5)				**(1,0),(0,2)**
13	**4.7124**	2.4837										
	(3,0)	(1,3),(3,1)										

2.4. Plate with a free edge

Boundary conditions at the free edge $x_1 = 0$ are

$$\sigma_{11} = \sigma_{21} = \sigma_{31} = 0. \tag{25}$$

For the in-plane modes of vibrations (7) and (9), $\sigma_{21} \neq 0$. Thus, these modes are not feasible in a rectangular plate with at least one edge surface traction free.

2.5. Cylindrical bending vibrations

For cylindrical bending vibrations in the x_1–x_3 plane, $u_2 \equiv 0$. For simply supported edges, boundary conditions

$$\sigma_{11} = 0, \quad u_3 = 0 \quad \text{on } x_1 = 0, \; L_1, \tag{26}$$

are satisfied by the displacement field (7). However, $L_2 = \infty$. Therefore, $u_1 = 0$ and we have a null solution. For clamped edges, boundary conditions

$$u_1 = 0, \quad u_3 = 0 \quad \text{on } x_1 = 0, \; L_1, \tag{27}$$

again give a null solution of the form (7). Thus, the assumption of cylindrical bending rules out in-plane mode of vibration (7).

3. Conclusions

We have delineated some of the in-plane modes of free vibration of a simply supported rectangular plate that were missed in previous exact solutions. For a homogeneous isotropic square plate with length equal to five times the thickness and the Poisson ratio = 0.3, the lowest frequency of an in-plane mode of vibration lies between the frequencies of the first two flexural modes.

Note added in proof

Liew et al. [14] used the Ritz method to analyze free vibrations of a thick rectangular plate. Their numerical solution gives accurate value of $\lambda_{10} = \lambda_{01}$ and $\lambda_{20} = \lambda_{02}$ for a square plate, and of λ_{10}, λ_{01} for a rectangular plate with $L_1/L_2 = 2$. However, Eqs. (6) through (9) cannot be deduced from their solution. Similar remarks apply to the solution of Qian et al. [15] computed by using the meshless local Petrov-Galerkin method. We have listed in Table 1 the first few frequencies of a square and a rectangular isotropic plate.

References

[1] S. Srinivas, A.K. Rao, Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates, International Journal of Solids and Structures 6 (1970) 1463–1481.

[2] S. Srinivas, C.V.J. Rao, A.K. Rao, An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates, Journal of Sound and Vibration 12 (1970) 187–199.
[3] P. Heyliger, D.A. Saravanos, Exact free-vibration analysis of laminated plates with embedded piezoelectric layers, Journal of the Acoustical Society of America 98 (1995) 1547–1555.
[4] R.C. Batra, X.Q. Liang, The vibration of a rectangular laminated elastic plate with embedded piezoelectric sensors and actuators, Computers and Structures 63 (1997) 203–216.
[5] R.C. Batra, X.Q. Liang, J.S. Yang, The vibration of a simply supported rectangular elastic plate due to piezoelectric actuators, International Journal of Solids and Structures 33 (1996) 1597–1618.
[6] I.S. Sokolnikoff, Mathematical Theory of Elasticity, McGraw-Hill, New York, 1956.
[7] P. Chadwick, M. Vianello, S.C. Cowin, A new proof that the number of linear elastic symmetries is eight, Journal of the Mechanics and Physics of Solids 49 (2001) 2471–2492.
[8] Z.Q. Cheng, R.C. Batra, Exact correspondence between eigenvalues of membranes and functionally graded simply supported polygonal plates, Journal of Sound and Vibration 229 (2000) 879–895.
[9] J.M. Whitney, N.J. Pagano, Shear deformation in heterogeneous anisotropic plates, Journal of Applied Mechanics 37 (1970) 1031–1036.
[10] J.N. Reddy, A simple higher-order theory for laminated composite plates, Journal of Applied Mechanics 51 (1984) 745–752.
[11] S. Vidoli, R.C. Batra, Derivation of plate and rod equations for a piezoelectric body from a mixed three-dimensional variational principle, Journal of Elasticity 59 (2000) 23–50.
[12] R.C. Batra, S. Vidoli, Higher-order piezoelectric plate theory derived from a three-dimensional variational principle, American Institute of Aeronautics and Astronautics Journal 40 (2002) 91–104.
[13] R.C. Batra, S. Vidoli, F. Vestroni, Plane waves and modal analysis in higher-order shear and normal deformable plate theories, Journal of Sound and Vibration 257 (2002) 63–88.

Additional references

[14] K.M. Liew, K.C. Hung, M.K. Lim, Three-dimensional vibration of rectangular plates: effects of thickness and edge constraints, Journal of Sound and Vibration 182 (1995) 709–727.
[15] L.F. Qian, R.C. Batra, L.M. Chen, Free and forced motions of a thick plate by using a higher-order shear and normal deformable plate theory and a meshless local Petrov-Galerkin method, Computer Modeling in Engineering and Sciences, in press.